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SEPARATING ULTRAFILTERS ON UNCOUNTABLE CARDINALS

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ABSTRACT

A uniform ultrafilter U on κ is said to be λ-separating if distinct elements of the ultrapower never project U to the same uniform ultrafilter V on λ. It is shown that, in the presence of CH, an ω-separating ultrafilter U on $\kappa > \omega$ is non-(ω, ω_1)-regular and, in fact, if $\kappa < \aleph_\omega$ then U is λ-separating for all λ. Several large cardinal consequences of the existence of such an ultrafilter U are derived.

§1. Introduction

We begin by establishing our notation and terminology. Throughout this paper κ, λ, μ etc. will denote infinite (but not necessarily regular) cardinals, and ${}^\kappa\lambda$ will denote the set of all functions mapping κ to λ. Suppose now that U is an ultrafilter on κ. U is said to be *uniform* if every set in U has cardinality κ. The usual equivalence relation $\sim_U$ on ${}^\kappa\lambda$ is given by $f \sim_U g$ iff $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$, and we let the equivalence class of f be denoted by $[f]_U$. The set of such equivalence classes can be linearly ordered by setting $[f]_U \leqq [g]_U$, iff $\{\alpha < \kappa : f(\alpha) \leqq g(\alpha)\} \in U$; the resulting structure is referred to as the *ultrapower of λ with respect to U.* If $f \in {}^\kappa\lambda$ then *f projects* U to an ultrafilter $f_*(U)$ on λ where $X \in f_*(U)$ iff $f^{-1}(X) \in U$. The ordering given by declaring $f_*(U) \leqq_{RK} U$ is called the *Rudin–Keisler ordering.* The property of ultrafilters that we will consider here is given by the following.

DEFINITION 1.1. Suppose that U is a uniform ultrafilter on κ and $\lambda \leqq \kappa$. Then U will be called *λ-separating* iff whenever $f_*(U)$ is a uniform ultrafilter on λ, the following implication holds:

$$\forall g \in {}^\kappa\lambda\,([f]_U \neq [g]_U \Rightarrow f_*(U) \neq g_*(U)).$$

U is said to be *separating* if U is λ-separating for every $\lambda \leqq \kappa$.

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The notion of a separating ideal was introduced in [10]; it is an easy exercise to show that an ultrafilter U is separating in the sense of Definition 1.1 iff the ideal on κ dual to U is separating in the sense of [10].

In Section 2, we consider non-regularity properties of separating ultrafilters and obtain some companion results to those of Pelletier [11]. In particular, we show that if U is an ω-separating ultrafilter on κ and CH holds, then U is non-(ω, ω_1)-regular, and if $\kappa < \aleph_\omega$ then U is non-(λ, λ^+)-regular for every $\lambda \leqq \kappa$. Several large cardinal consequences of the existence of a separating ultrafilter are discussed in Section 3. In Section 4, we show that if U is λ-separating and non-(λ, λ^+)-regular, then U is λ^+-separating; this result is reminiscent of the well-known analogous result for λ-descendingly incomplete ultrafilters [3], [4], [9].

§2. Non-regularity properties of separating ultrafilters

Recall that a uniform ultrafilter U on κ is said to be *(λ, μ)-regular* iff there are μ sets in U any λ of which have empty intersection. Such a collection is called a *(λ, μ)-regularizing family for U.* If U fails to be (ω, κ)-regular, then U is said to be *non-regular.* Pelletier was the first to point out that separating ultrafilters possess a degree of non-regularity; his method of proof yields the following (although only a special case is explicitly stated in [11]).

THEOREM 2.1 (Pelletier [11]). *Suppose that U is a separating ultrafilter on κ and that γ is a cardinal satisfying:*

$$2^{2^{2^{<\gamma}}} < 2^{\kappa}.$$

Then U is non-(γ, κ)-regular.

The above result, however, yields no information for the case $\kappa = \omega_1$. Thus, we take another approach to irregularity. This approach requires the following three lemmas, the first of which combines ideas of Blass [2] p. 34, Benda–Ketonen [1], and Jorgensen [6].

LEMMA 2.2. *Suppose that U is an $(\omega, 2^\lambda)$-regular uniform ultrafilter on κ and V is an arbitrary uniform ultrafilter on λ. Then there are $(2^\lambda)^+$ distinct elements of the ultrapower, all of which project U onto V.*

PROOF. Let $\{A_\alpha : \alpha < 2^\lambda\}$ be an $(\omega, 2^\lambda)$-regularizing family for U and let $\{X_\alpha : \alpha < 2^\lambda\}$ be an enumeration of the sets in V. It clearly suffices to show that for any collection $\{f_\alpha : \alpha < 2^\lambda\}$ of functions mapping κ to λ, we can find a function $f : \kappa \to \lambda$ so that

(a) $[f_\alpha] \neq [f]$ for every $\alpha < 2^\lambda$, and

(b) $f_*(U) = V$.

We will accomplish this by constructing $f : \kappa \to \lambda$ so that

(a′) $f_\alpha(\xi) < f(\xi)$ for every $\xi \in A_\alpha$ and $\alpha < 2^\lambda$, and

(b′) $f^{-1}(X_\alpha) \supseteq A_\alpha$ for every $\alpha < 2^\lambda$.

For each $\xi < \kappa$, let $\mathcal{O}(\xi) = \{\alpha < 2^\lambda : \xi \in A_\alpha\}$. Since infinite intersections of the A_α's are empty, we know that $\mathcal{O}(\xi)$ is finite. Hence, if we let $X = \bigcap\{X_\alpha : \alpha \in \mathcal{O}(\xi)\}$, then $|X| = \lambda$ and so we can choose $f(\xi) \in X$ so that for every $\alpha \in \mathcal{O}(\xi)$ we have $f_\alpha(\xi) < f(\xi)$. Notice that

(a″) if $\alpha < 2^\lambda$ and $\xi \in A_\alpha$ then $\alpha \in \mathcal{O}(\xi)$ so $f_\alpha(\xi) < f(\xi)$, and

(b″) if $\xi \in A_\alpha$ then $\alpha \in \mathcal{O}(\xi)$ so $f(\xi) \in X_\alpha$.

Since (a″)→(a′)→(a) and (b″)→(b′)→(b), the proof is complete.

The next lemma is again heavily based on ideas of Benda–Ketonen [1]; its statement is aided by the following bit of terminology.

DEFINITION 2.3. If U is a uniform ultrafilter on κ, then $\mathcal{F}$ will be called a *λ-family for U* iff $\mathcal{F}$ consists of functions each mapping a set in U to λ so that if $f, g \in \mathcal{F}$ and $f \neq g$ then

$$|\{\xi \in \mathrm{domain}(f) \cap \mathrm{domain}(g) : f(\xi) = g(\xi)\}| < \kappa.$$

LEMMA 2.4. *Suppose that U is a uniform $(\lambda^+, \lambda^{++})$-regular ultrafilter on κ, and assume that there is a λ^+-family for U of size λ^{++}. Then U is (λ, λ^+)-regular.*

PROOF. Let $\{A_\alpha : \alpha < \lambda^{++}\}$ show that U is $(\lambda^+, \lambda^{++})$-regular and let $\{f_\alpha : \alpha < \lambda^{++}\}$ be a λ^+-family for U where $f_\alpha : X_\alpha \to \lambda^+$. Define $g : \kappa \to \lambda^+$ so that if $\xi \in A_\alpha$ then $f_\alpha(\xi) < g(\xi)$. This is possible since ξ occurs in only λ many A_α's. For each $\gamma < \lambda^+$ let $h_\gamma : \gamma \to \lambda$ be one to one and for each $\alpha < \lambda^{++}$ let $f'_\alpha : A_\alpha \to \lambda$ be given by $f'_\alpha(\xi) = h_{g(\xi)}(f_\alpha(\xi))$. Notice that $\{f'_\alpha : \alpha < \lambda^{++}\}$ is a λ-family for U. Without loss of generality, assume that for each $\alpha < \lambda^+$ there is a set $B_\alpha \in U$ so that $f'_\alpha(\xi) < f_{\lambda^+}(\xi)$ for every $\xi \in B_\alpha$. Finally, let $C_\alpha \in U$ be given by $C_\alpha = B_\alpha - \{\xi < \kappa : \exists \beta < \alpha (f'_\beta(\xi) = f'_\alpha(\xi))\}$. It is easy to see that $\{C_\alpha : \alpha < \lambda^+\}$ is a (λ, λ^+)-regularizing family for U.

The non-regularity results for separating ultrafilters that follow from Lemmas 2.2 and 2.4 are summarized in the following.

THEOREM 2.5. *Suppose that U is a uniform ultrafilter on κ.*

(a) *If U is λ-separating, then U is non-$(\omega, 2^\lambda)$-regular.*

(b) (CH). *If U is ω-separating then U is non-(ω, ω_1)-regular; in particular, U is non-regular.*

(c) (CH). *If* $\kappa < \aleph_\omega$ *and* U *is* ω*-separating then* U *is non-*(λ, λ^+)*-regular for every* λ.

PROOF. Parts (a) and (b) are immediate from Lemma 2.2. Part (c) follows from part (b), Lemma 2.4, and the observation that if $\kappa < \aleph_\omega$ and $\lambda < \kappa$ then there is a λ^+-family for U of size λ^{++}. (One starts with a family of κ^+ eventually different functions from κ to κ, i.e. the case $\lambda = \kappa^-$, and then works one's way down to λ using the same argument that occurred in the proof of Lemma 2.4.)

§3. Large cardinal consequences

An ultrafilter U on κ is said to be *weakly normal* iff whenever $\{\alpha < \kappa : f(\alpha) < \alpha\} \in U$, there is a $\beta < \kappa$ so that $\{\alpha < \kappa : f(\alpha) \leqq \beta\} \in U$. U is said to be λ*-indecomposable* iff there is *no* uniform ultrafilter V on λ such that $V \leqq_{RK} U$. Notice that if U is λ-indecomposable then U is λ-separating. The large cardinal consequences of the existence of a separating ultrafilter on κ that we obtain in this section are derived from the following well-known results.

THEOREM 3.1 (a) (Kanamori [7]). *If there is a uniform non-*(κ, κ^+)*-regular ultrafilter* U *on* κ^+, *then there is such an ultrafilter* V *on* κ^+ *which is also weakly normal and less than or equal to* U *in the Rudin–Keisler ordering.*

(b) (Kanamori [7] and Ketonen [8] independently). *If there is a uniform ultrafilter* U *on a regular cardinal* κ *which in non-*(ω, λ)*-regular for some* $\lambda < \kappa$, *then there is such an ultrafilter* V *on* κ *which is also weakly normal.*

(c) (Jensen [5]). *Suppose that* $\kappa^{<\kappa} = \kappa$ *and there is a uniform weakly normal ultrafilter on* κ. *Then there is an inner model with a measurable cardinal.*

(d) (Koppelberg for regular κ [5]; Donder for singular κ). *Suppose that there is a uniform ultrafilter on* κ *which is* λ*-indecomposable for some regular* $\lambda < \kappa$. *Then there is an inner model with a measurable cardinal.*

The following is now straightforward.

THEOREM 3.2. *Suppose that* U *is an* ω*-separating ultrafilter on* $\kappa > \omega$, *and either*

(i) CH *holds, or*

(ii) $\kappa > 2^{\aleph_0}$ *and* $\kappa^{<\kappa} = \kappa$.

Then there is an inner model with a measurable cardinal.

PROOF. Suppose first that (i) holds. Then either U is ω_1-indecomposable, in which case we are done by Theorem 3.1(d), or there is a uniform ultrafilter V on ω_1 with $V \leqq_{RK} U$. It is an easy exercise to show that in this case V is also

ω-separating and, hence, non-regular by Theorem 2.5(c). But now we are done by Theorem 3.1(a) and (c).

If (ii) holds, then U is non-(ω, λ)-regular for $\lambda = 2^{\aleph_0} < \kappa$ by Theorem 2.5(a). The desired result now follows from Theorem 3.1(b) and (c).

This is the best possible result on the consistency strength of the existence of a separating ultrafilter on some $\kappa > \omega$, except in cases like $\kappa \leqq 2^{\omega}$. When κ is strongly inaccessible, the following result shows that κ itself has substantial large cardinal properties.

THEOREM 3.4. *Suppose that U is a separating ultrafilter on the strongly inaccessible cardinal κ. Then*:

(a) *κ is in the ωth strong Mahlo class.*

(b) *If the* GCH *holds below κ, then $2^{\kappa} = \kappa^{+}$.*

(c) *Kurepa's Hypothesis for κ fails.*

PROOF. The proofs amount to a recasting of results in [12]. For (a), note first that by 2.5(a) and 3.1(b) we can assume that U is weakly normal. Moreover, it is easy to see that $|{}^{\kappa}\gamma/U| < \kappa$ for every $\gamma < \kappa$; i.e. if $f, g \in {}^{\kappa}\gamma$ and $[f]_U \neq [g]_U$ then $f_*(U) \neq g_*(U)$, and there are fewer than κ many ultrafilters on γ. By straightforward arguments (see proposition 8 of [12]) this is enough to verify that $\{\alpha < \kappa : \alpha$ is strongly inaccessible$\}$ is in U. We can now proceed by induction to establish that for each $n \in \omega$, $\{\alpha < \kappa : \alpha$ is nth-strongly Mahlo$\} \in U$. This is achieved by following the proof of theorem 6 of [12], using for the 1st case in that proof the fact that if $V \leqq_{\mathrm{RK}} U$, then V is also separating.

For (b), we again assume that U is weakly normal and $|{}^{\kappa}\gamma/U| < \kappa$ for every $\gamma < \kappa$ and call upon the proof of theorem 16 of [12]; this argument is essentially Scott's proof that if V is a normal ultrafilter on a measurable cardinal μ and $\{\alpha < \mu : 2^{\alpha} = \alpha^{+}\} \in V$, then $2^{\mu} = \mu^{+}$.

Finally, (c) follows in analogous fashion from theorem 7 of [12].

Whilst on the topic of large cardinals, let us mention a result of Sureson (unpublished). A normal ultrafilter on a measurable cardinal is separating, so it is natural to ask whether being a p-point, a well-known property of ultrafilters weaker than normality, is also a sufficient condition. Sureson established that this is not so. Specifically, she established that if κ is $2^{2^{\kappa}}$-supercompact (sic), then there is a p-point on κ which is not separating. Sureson has also shown that the consistency of the existence of a measurable cardinal is enough to obtain the consistency of the existence of a measurable cardinal which carries a non-separating p-point ultrafilter.

§4. A stepping up theorem

It is well-known that if λ is regular and U is a λ-indecomposable ultrafilter, then U is also λ^+-indecomposable. (This was first proved by Chang [3] assuming $2^\lambda = \lambda^+$ and in general by Chudnovsky and Chudnovsky [4] and Kunen and Prikry [9].) The following result provides a partial analogue of this property for λ-separating ultrafilters.

THEOREM 4.1. *Suppose that λ is regular and that U is λ-separating and non-(λ, λ^+)-regular. Then U is λ^+-separating.*

PROOF. Assume that U is a uniform ultrafilter on κ and that $f, g : \kappa \to \lambda^+$ show that U is not λ^+-separating. We want to show that U is either (λ, λ^+)-regular or not λ-separating. For this, we will need the following lemmas.

LEMMA 4.2. *There exists a collection $\{f_\alpha : \alpha < \lambda^+\}$ of functions satisfying the following*:

(i) *for each $\alpha < \lambda^+$, $f_\alpha : |\alpha| \to \alpha$ is a bijection, and*

(ii) *if $\beta < \alpha < \lambda^+$ then $|\{\xi < \lambda : f_\beta(\xi) = f_\alpha(\xi)\}| < \lambda$.*

PROOF. For $\alpha < \lambda$, choose any f_α satisfying (i). Suppose now that $\lambda \leqq \alpha < \lambda^+$ and that f_β has been defined for each $\beta < \alpha$. Let $\{g_\xi : \xi < \lambda\}$ enumerate $\{f_\beta : \beta < \alpha\}$ in order-type λ and let $\{\gamma_\xi : \xi < \lambda\}$ enumerate α in order-type λ. We will define a bijection $f_\alpha : \lambda \to \alpha$ by a back and forth induction involving λ steps, where at step $\xi < \lambda$ we specify values for $f_\alpha(\xi)$ and $f_\alpha^{-1}(\gamma_\xi)$. In order to ensure that (i) and (ii) hold we need only do this so that f_α remains one to one and the following are satisfied:

(iii) if $\eta \leqq \xi$ and $f_\alpha(\xi)$ has not yet been defined then $f_\alpha(\xi) \neq g_\eta(\xi)$;

(iv) if $\eta \leqq \xi$ and $f_\alpha^{-1}(\gamma_\xi)$ has not yet been defined then $f_\alpha^{-1}(\gamma_\xi) \neq g_\eta^{-1}(\gamma_\xi)$.

It is easy to see that this is possible. To see that (ii) holds notice that if $\eta < \lambda$ and $f_\alpha(\xi) = g_\eta(\xi) = \gamma_{\eta'}$, then $\xi < \max\{\eta, \eta'\}$; i.e. if $f_\alpha(\xi)$ was defined at stage ξ and $\xi \geqq \eta$ then $f_\alpha(\xi) \neq g_\eta(\xi)$ by (iii) and if $f_\alpha(\xi)$ was defined at stage $\eta' < \xi$ then $f_\alpha^{-1}(\gamma_{\eta'}) \neq g^{-1}(\gamma_{\eta'})$ by (iv).

Now, to complete the proof of Theorem 4.1 we define, for each $\alpha < \lambda^+$, a function $h_\alpha : \lambda^+ - (\alpha + 1) \to \lambda$ by

$$h_\alpha(\beta) = f_\beta^{-1}(\alpha).$$

Recall that $f, g : \kappa \to \lambda^+$ were chosen so that $[f]_U \neq [g]_U$ but $f_*(U)$ and $g_*(U)$ are the same uniform ultrafilter on λ^+. Without loss of generality, assume that $f(\xi) < g(\xi)$ for every $\xi < \kappa$. We consider 3 cases.

Case 1. $\{\alpha < \lambda^+ : (h_\alpha \circ f)_*(U)$ is not uniform on $\lambda\}$ has cardinality λ^+.

In this case we get a cardinal $\mu < \lambda$, a set $Z \subseteq \lambda^+$ and for each $\alpha \in Z$ a set $X_\alpha \in U$ so that $|Z| = \lambda^+$ and $h_\alpha(f(X_\alpha)) \subseteq \mu$. Let $Y_\alpha = X_\alpha - \{\gamma < \kappa : f(\gamma) \leqq \alpha\}$. Notice that $Y_\alpha \in U$ since $f_*(U)$ is a uniform ultrafilter on λ^+. We claim that $\{Y_\alpha : \alpha < \lambda^+\}$ shows that U is (λ, λ^+)-regular. To see this, suppose not and choose γ occurring in λ many Y_α's. Let $\beta = f(\gamma)$. Since $h_\alpha(\beta) < \mu$ we get a set $A \subseteq \lambda^+$ so that $|A| = \lambda$ and for each $\alpha, \alpha' \in A$ we have $h_\alpha(\beta) = h_{\alpha'}(\beta)$. (Notice that for each such α we have $h_\alpha(\beta)$ defined since $\gamma \in Y_\alpha \to f(\gamma) > \alpha \to \beta > \alpha$. Thus $\alpha < \beta$ so $\beta \in \mathrm{domain}(h_\alpha)$.) But now we have $f_\beta^{-1}(\alpha) = f_\beta^{-1}(\alpha')$, contradicting the fact that f_β is one to one.

Case 2. $\{\alpha < \lambda^+ : [h_\alpha \circ f]_U = [h_\alpha \circ g]_U\}$ has cardinality λ^+.

Let Z be the set of such α and choose $X_\alpha \in U$ for each $\alpha \in Z$ so that $h_\alpha \circ f(\gamma) = h_\alpha \circ g(\gamma)$ for every $\gamma \in X_\alpha$. We claim that the collection $\{X_\alpha : \alpha \in Z\}$ shows that U is (λ, λ^+)-regular. To see this, suppose not and choose γ occurring in λ many X_α's. Then for each such α we have $f_{f(\gamma)}^{-1}(\alpha) = f_{g(\gamma)}^{-1}(\alpha)$ and so $f_{f(\gamma)}^{-1}$ and $f_{g(\gamma)}^{-1}$ agree on a set of size λ. Thus $f(\gamma) = g(\gamma)$, contradiction.

Case 3. Otherwise.

In this case we have at least one h_α so that

$$[h_\alpha \circ f]_U \neq [h_\alpha \circ g]_U$$

and $(h_\alpha \circ f)_*(U)$ is a uniform ultrafilter on λ. Since $f_*(U) = g_*(U)$ it follows that $(h_\alpha \circ f)_*(U) = (h_\alpha \circ g)_*(U)$ and so U is not λ-separating in this case.

Combining Theorem 4.1 with the non-regularity results in Theorem 2.5(b) and (c), we obtain the following.

THEOREM 4.3 ([3]). *Assume that U is an ω-separating ultrafilter on κ. Then*
(a) *U is ω_1-separating, and*
(b) *if $\kappa < \aleph_\omega$, then U is a separating ultrafilter (i.e., λ-separating for all λ).*

It is worth noting that the converse of Theorem 4.3(a) is not provable. In fact, the existence of an ω_1-separating ultrafilter on ω_1 has no large cardinal consequences. For example, if $2^{\omega_1} = \omega_2$, then a straightforward inductive construction yields a uniform ultrafilter U on ω_1 having the property that any $f : \omega_1 \to \omega_1$ is either bounded (mod U) or one to one (mod U). (This was pointed out to us several years ago by Prikry.) But, as shown in [10], *every* ideal (in particular: U^*) is separating with respect to one-one functions, and so U is ω_1-separating.

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